HYPERSPHERIC WAVE FRONT

A. Rodríguez Carassus

This collection of articles published on **ResearchGate**, develops the **HYPERSPHERIC WAVE FRONT theory**, which proposes the idea of reinterpreting our three-dimensional universe as a hyperspherical wavefront expanding in a linear fourth dimension at the speed of light.
Attached are:

-HYPERSPHERIC WAVE FRONT (2022).

-HYPERSPHERIC WAVE FRONT II: "Additional considerations related to mass" (2022).

-HYPERSPHERIC WAVE FRONT III: "Vector Product in Hyperspace" (2024).

- HYPERSPHERIC WAVE FRONT IV: "Complex Hyperspace C^7" (2025).

HYPERESPHERIC WAVE FRONT

Essay written for the Gravity Research Foundation 2022 Awards for Essays on Gravitation

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ABSTRACT:

Three arguments will be provided compatible with the reinterpretation of our three dimentional universe, like a hyperespheric wavefront, expanding itself through a fourth lineal dimention at the speed of light.
Its viability would open the possible reconsideration of gravity and the rest fundamental forces, in a new theoretical framework.

January 14, 2022

INTRODUCTION:

Preliminary explanation of the hypothesis:

When interpreting our three dimensional universe as a hyperspheric wavefront, we are considering a three dimensional spherical surface, which expands in a four dimensional Cartesian space, that is, one that admits a maximum of four linearly independent vectors belonging to it.
It is also suggested that this wavefront expands at the speed of light, therefore, the radius of this hyperspheric wavefront grows at the speed of light.

Time redefinition:

The development of the arguments that justify the proposed hypothesis, requires the redefinition that we will use in the present work, of a fundamental variable: time.
We will use a "light clock" in which we will measure the distance traveled by the light, as a temporal evaluation measure for an event in a given reference frame. It is a well known resource, used especially in the study of problems related to Relativity.
Our vision from now on will be subtly different, we will define the time associated with an event in a given reference frame, as the distance traveled by light simultaneously to that event in said reference.
We do not presuppose the existence of a concept of time, but that now time is a distance traveled by light, and therefore has a dimension of length.
We will designate the distances with the variable x and the times with the variable λ.
Both quantities have dimensions of length

$$\dim x = L \tag{1}$$

$$\dim \lambda = L \tag{2}$$

If we consider differential displacements dx and dλ we will have a situation like the one considered in Figure 1.
The point P at the initial instant is at the origin of coordinates and after a period of time it is at a certain distance dx from O, in the direction marked by the versor **i**.
At the initial instant, time begins to be measured by means of a light clock that will read dλ, which is the radius of a spherical wave centered in O synchronized with the arrival of P at the end point of the movement, that is, after traveling dx .

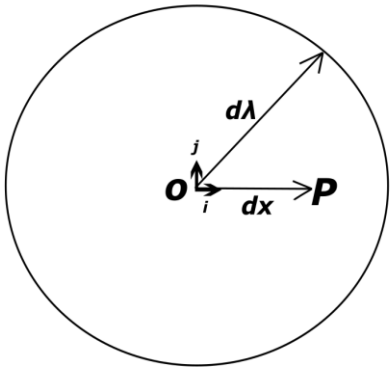

Figure 1

We will place a superscript (*) to the magnitudes or variables **m**, calculated taking the definition of time considered, and they will be transformed into **m***.
The velocity u* of P is defined by the quotient:

$$u^* = \frac{dx}{d\lambda} \tag{3}$$

Therefore u * is a dimensionless quantity

$$\dim u^* = 1 \tag{4}$$

The relativistic postulate of constancy of the speed of light applies for all inertial systems, but its value, when time is defined as the distance traveled by light is not the constant c, whose approximate value is c=3 x10^8 m/s, but c*=1.
Note that from the operational point of view, to transform any quantity defined by an equation with our redefinition of time dλ, we substitute: dt for dλ, c for c*=1, the velocities go from v to v * and the lengths remain equal.

$$c = \frac{d\lambda}{dt} \tag{5}$$

$$c^* = \frac{d\lambda}{d\lambda} = 1 \tag{6}$$

For the general case of speeds, the conventional definition being

$$u = \frac{dx}{dt} \tag{7}$$

It will then be

$$u^* = \frac{dx}{d\lambda} \tag{8}$$

an equivalence is reached

$$u^* = \frac{u}{c} \tag{9}$$

We are now going to raise the traditional experiment in relativity, by which time dilation is explained.
We recall that in the same two systems S and S' whose origins O and P coincide in the initial instant are considered.
S is attached to a fixed platform on which a rail car travels with speed u to which S' is attached.

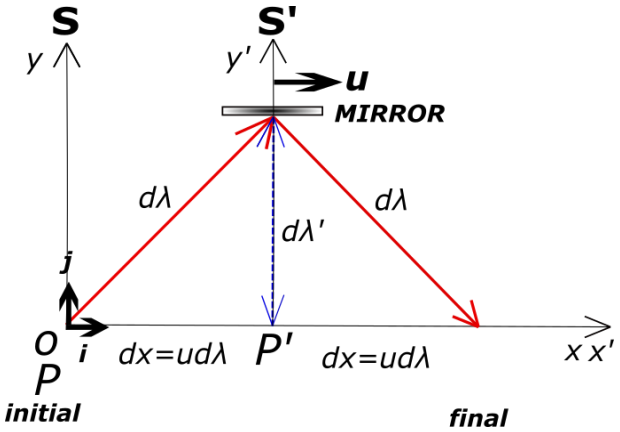

Figure 2

At the initial instant, a ray is thrown at the mirror fixed at S', as indicated in Figure 2.
We will find using the Pythagorean theorem the relationship of dλ with dλ '.

Being γ by definition

$$\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \tag{10}$$

By Equations 6 and 9 we have that

$$\gamma^* = \frac{1}{\sqrt{1-u^{*2}}} \tag{11}$$

$$\gamma^* = \gamma \tag{12}$$

$$d\lambda^2 = d\lambda'^2 + dx^2 \tag{13}$$

$$dx = u^* d\lambda \tag{14}$$

$$d\lambda = \frac{d\lambda'}{\sqrt{1-u^{*2}}} \tag{15}$$

$$d\lambda = \gamma d\lambda' \tag{16}$$

Thus, the rapid visualization and understanding of the time relationship is wiewed, as well as its totally graphic approach, a practice that will be of great help when advancing in the development of this work.

In the same way, the relation of the lengths l in S and l' in S' is

$$l = \frac{l'}{\gamma^*} \tag{17}$$

Let us now recall the Lorentz transformations, that allow us to relate the times and distances in two inertial systems S and S' that evaluate the same event, for example the displacement of point P, and where S' moves with velocity u with respect to S.

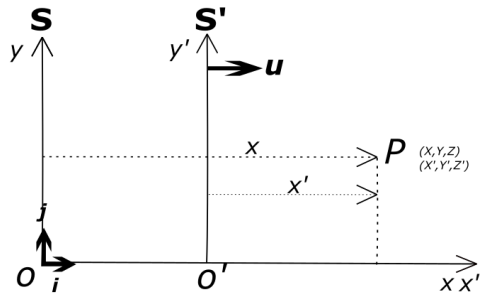

Figure 3

In the case of expressing the equations as a function of the redefinition of the time λ considered, they will remain:

$$x' = \gamma^*(x - u^*\lambda) \tag{18}$$

$$y' = y \tag{19}$$

$$z' = z \tag{20}$$

$$\lambda' = \gamma^*(\lambda - u^* x) \tag{21}$$

With these preliminary considerations established, we will proceed to develop the three arguments.

1: The hypothesis proposed is compatible with the expansion speed of the Universe derived from Hubble's Law.

According to our hypothesis, our universe is a three dimensional hypersphere that expands radially at the speed of light.
The data derived from Hubble's law, show an approximate age of the universe of 14,000 million years, and an approximate value of its expansion speed of 70 km/sec per Mpc (Megaparsec).
Based on our definition of time, if the age of the universe is 14 billion years, the radius of the hypersphere that we call λ_0, must meet

$$\lambda_0 = 14x10^9 light - years \tag{22}$$

Being the radial velocity of growth the speed of light c*=1, we will find the tangential velocity of expansion of a 1Mpc arc, which we will call V*e.

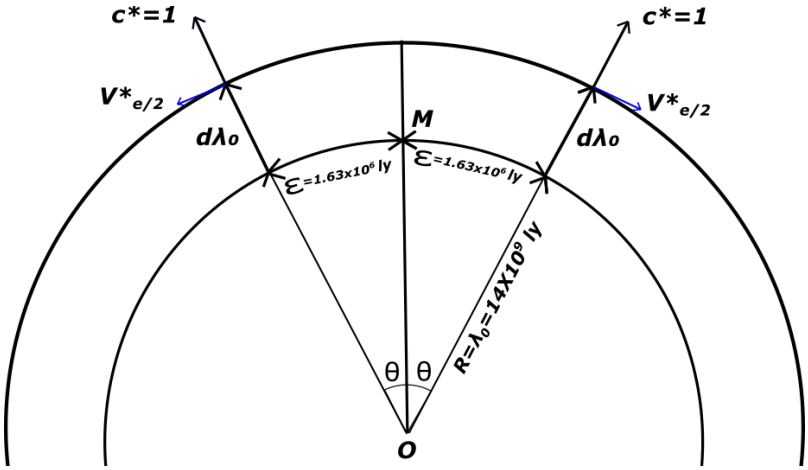

Figure 4

Figure 4 graphically represents the problem considered, the radius of our hypersphere, which we call R, is equivalent to the total expansion of our universe from the initial moment or Big Bang.
We divide the 1 Mpc arc into two equal semi-arcs that we will call ε.
We will call V*e/2 to the middle of V*e, and it will be measured with respect to the point M located in the middle of the arc, and considered fixed with respect to the tangential displacement.

$$R = \lambda_0 \tag{23}$$

Now let us consider a differential displacement of our wavefront in the fourth dimension $d\lambda_0$.
A Megaparsec is by definition 3.26 million light-years length, therefore for our hypothesis, it is an arc of the hypersphere of 3.26 million light-years.
Therefore, each semi-arch that we call ε, has a length of 1.63 x 10^6 ly and subtend an angle that we will call θ, (that we will express in radians).
It then follows that:

$$\theta = \frac{\varepsilon}{R} = \frac{1.63\times10^6 ly}{14\times10^9 ly} \tag{24}$$

$$\theta == 1.1643 \times 10^{-4} \tag{25}$$

$$V^*_{e/2} = \frac{(R + d\lambda_0)\theta - R\theta}{d\lambda_0} \tag{26}$$

$$V^*_{e/2} = \theta \tag{27}$$

$$V^*_e = 2V^*_{e/2} = 2\theta \tag{28}$$

$$V^*_e = 2\theta = 2 \times 1.1643 \times 10^{-4} \tag{29}$$

$$V^*e = 2.3286 \times 10^{-4} \tag{30}$$

It follows then that if we express Ve in kilometers per second:

$$V_e = 2.3286 \times 10^{-4} c = 2.3286 \times 10^{-4} \times 3 \times 10^5 \, Km/seg \tag{31}$$

$$Ve = 69.86 \, Km/seg \tag{32}$$

We see that this value agrees very closely with the data from the 2006 NASA WMAP probe, which yielded an approximate value of Ve =70 km/s per Mpc.

2: The proposed hypothesis is geometrically compatible with the relationships between distances and times derived from the Lorentz Transformations.

Let us initially consider the motion of a point P, seen from the third dimension, as described in Figures 1 and 2.

Rearranging equation (13) we have

$$d\lambda'^2 = d\lambda^2 - dx^2 \tag{33}$$

Let us observe that it expresses the value of the proper time of P, as a function of the time and the distance traveled with respect to O, which are dλ and dx respectively.

On the other hand, the Lorentz transformations establish the validity of the expression

$$ds^2 = d\lambda^2 - dx^2 \tag{34}$$

From the comparison of (33) and (34) we obtain the identity

$$ds = d\lambda' \tag{35}$$

This indicates that the invariant ds of the Lorentz transformations is equivalent to the proper time of the moving system.

Let's look at equation (34) again, the same is true for every point P that moves away from O with constant velocity. If said velocity is equal to zero and therefore dx = 0 we will have

$$d\lambda_0 = ds \tag{36}$$

We represent the advance of the Wave Front in Figure 5.

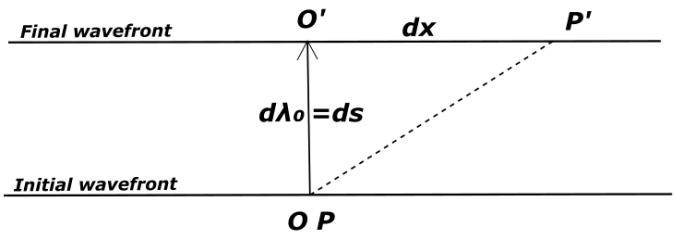

Figure 5

In the same, given the enormous magnitude of the radius of the Wavefront, R = 14 x 10^9 with respect to dx and $d\lambda_0$, it can be considered that it is flat for the purposes of the corresponding analysis.
At the initial instant, O and P coincide in the initial configuration of the Wavefront.
At the final instant, the Wavefront advanced $d\lambda_0$ while P traveled dx, which, since it is perpendicular to the movement of the front, has the same length with respect to the fourth dimension.
As a consequence of the above, the segment OP 'is equal to dλ (applying Pythagoras to the triangle OO'P').
As a graphic summary of these conclusions, in Figure 6, a system of Cartesian axes centered on P, has been superimposed on the advance of the wavefront in Figure 5, with the proper time of P on the ordinate and the distance x on abscissa.
The equation of the hyperbola H is

$$ds^2 = d\lambda^2 - dx^2 = d\lambda_i^2 - dx_i^2 \tag{37}$$

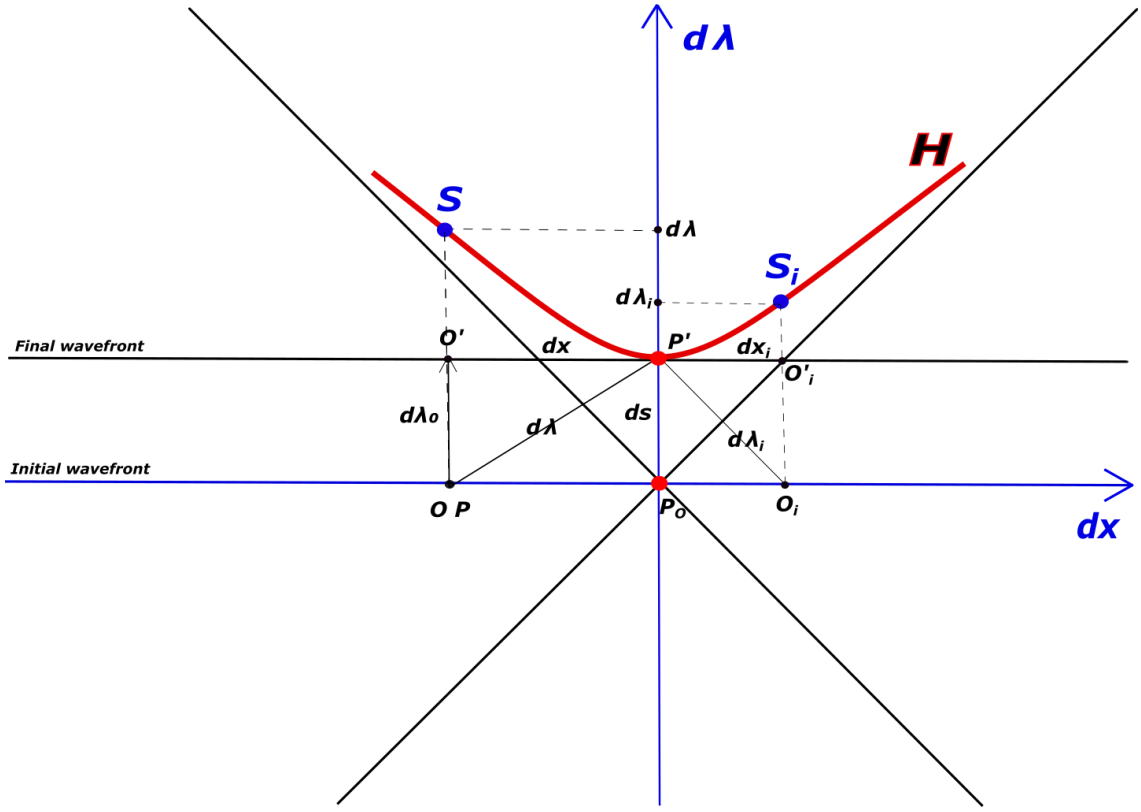

Figure 6

3): The proposed hypothesis is compatible with a reinterpretation of the electrostatic force as the magnetic force resulting from the displacement of electric charges, at the speed of light in the fourth dimension.

For the purposes of developing this argument, we will previously review the magnetic force F, which acts between two electric charges q with equal signs moving at equal and parallel speeds v, as shown in figure 7.
B is the magnetic field in the charge on the right, created by the charge on the left.

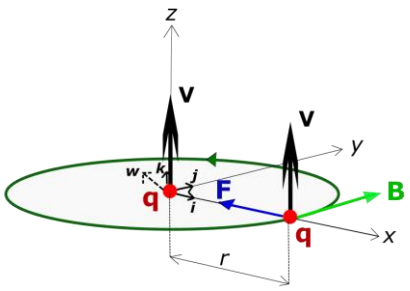

Figure 7

The force F the resultant of the vector product of qv and B, and is the magnetic force of attraction between the charges wich direction is X:

$$\vec{F} = -\frac{\mu_0 q^2 v^2}{4\pi r^2}\vec{i} \tag{38}$$

Vectors B, qv, and F are three orthonormal axis.
The dotted **w** versor represents a fourth axis orthonormal to the orthonormal basis **ijk**.
We will try to study by analogy, how the situation represented in Figure 7, should be transformed, if we observe it from the fourth dimension.
In the first place, the speeds of displacement of the charges are parallel and of the same sense and magnitude c, because it is an advance of the wave front and due to the equality of the sign of the charges.
We observe in Figure 7 that the magnetic field B generated by the movement of the left charge is tangent to a circumference of radius r, and the magnetic force of attraction F is in the direction of the line joining the charges.
What then should the equivalent situation look like when viewed from a higher dimension?
First of all, we understand that the circumference of radius r, must be transformed into a spherical surface of radius r centered on the charge on left. In turn, the field B should, by analogy, be contained in the plane tangent to it that passes through the charge q on the right.
We can make the Y axis coincide with the direction of B. On the other hand, F must be orthonormal to B and c, and placed in the direction of the line that joins the charges, that is the X axis.

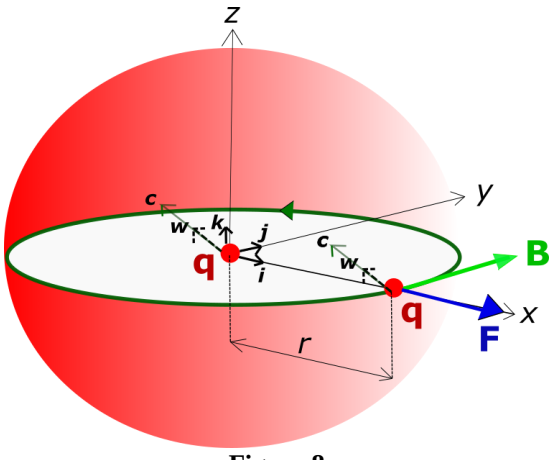

Figure 8

The sense remains to be determined.

We are faced with the difficulty that the vector product of 2 vectors (between the velocity v, in this case c and B) is not defined for four dimensions, since now the basis to consider should be **ijkw** for which the vectors should be expressed with four coordinates.
To solve this impossibility, we will resort to a property of imaginary numbers.
We remember that an imaginary number is of the form λi, where λ is a real number and i is the imaginary unit $i = \sqrt{-1}$.
The direction of an imaginary modulus vector is considered orthonormal to the real axes, that is, X, Y, Z.
Therefore, if in the formula (38) we do, we have:

$$\vec{F} = -\frac{\mu_0 q^2 (ci)^2}{4\pi r^2}\vec{i} \tag{39}$$

We obtain

$$\vec{F} = \frac{\mu_0 q^2 c^2}{4\pi r^2}\vec{i} \tag{40}$$

Which is the expression of the electrostatic force of repulsion between charges of the same sign.

We are going to raise a different point of view to analyze the same problem.
The main problem, as mentioned previously, is that the vector product is defined for two vectors qv and B in three dimensional space, and implies the resolution of the following determinant:

$$q\vec{v} = (qv_1, qv_2, qv_3) \tag{41}$$

$$\vec{B} = (B_1, B_2, B_3) \tag{42}$$

$$q\vec{v} \times \vec{B} = \begin{bmatrix} i & j & k \\ qv_1 & qv_2 & qv_3 \\ B_1 & B_2 & B_3 \end{bmatrix} \tag{43}$$

In case of calculate the vector product of two vectors qv and B of four dimensions, the resolution of an impossible determinant would be presented:

$$q\vec{v} \times \vec{B} = \begin{bmatrix} i & j & k & w \\ qv_1 & qv_2 & qv_3 & qv_4 \\ B_1 & B_2 & B_3 & B_4 \end{bmatrix} \tag{44}$$

We will propose an alternative possibility, that of solving the 3x3 sub-determinants that are achieved by eliminating a column.
Substituting the vectors qv and B by their components, with the value of the speed of light $v = ci$ we have

$$q\vec{v} \times \vec{B}(i, j, k, w) = \begin{bmatrix} i & j & k & w \\ 0 & 0 & 0 & qci \\ 0 & \frac{\mu_0 qci}{4\pi r^2} & 0 & 0 \end{bmatrix} \tag{45}$$

Being the sub-determinants to solve:

$$q\vec{v}\times\vec{B}(j,k,w)=\begin{bmatrix} j & k & w \\ 0 & 0 & qci \\ \frac{\mu_0 qci}{4\pi r^2} & 0 & 0 \end{bmatrix}=\frac{\mu_0 q^2 c^2}{4\pi r^2}\vec{k} \tag{46}$$

$$q\vec{v}\times\vec{B}(i,k,w)=\begin{bmatrix} i & k & w \\ 0 & 0 & qci \\ 0 & 0 & 0 \end{bmatrix}=0 \tag{47}$$

$$q\vec{v}\times\vec{B}(i,j,w)=\begin{bmatrix} i & j & w \\ 0 & 0 & qci \\ 0 & \frac{\mu_0 qci}{4\pi r^2} & 0 \end{bmatrix}=\frac{\mu_0 q^2 c^2}{4\pi r^2}\vec{i} \tag{48}$$

$$q\vec{v}\times\vec{B}(i,j,k)=\begin{bmatrix} i & j & k \\ 0 & 0 & 0 \\ 0 & \frac{\mu_0 qci}{4\pi r^2} & 0 \end{bmatrix}=0 \tag{49}$$

We see two non-null and equal results, (46) and (48), which is logical since the vector product of two vectors must be orthonormal to the factors, and **i** and **k** fulfill this condition against **j** and **w**.
Logically we discard the result according to **k**, because for experimental reasons we know that the force is according to **i**.
And we confirm by this proposed procedure, the result of the force expressed by equation (38).

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HYPERSPHERIC WAVE FRONT II
Additional considerations related to mass

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ABSTRACT:

The original essay, offered three arguments compatible with reinterpreting our three-dimensional universe as a hyperspheric wave front, expanding into a fourth linear dimension at the speed of light.
In the present work, the implications of the same regarding the mass of the bodies, are considered.

November 19, 2022

1. Momentum:

We will consider, using the hypotheses developed in the previous work entitled "HYPERSPHERIC WAVE FRONT", deduce the expression of the momentum for the same body with respect to two inertial systems that we will call S and S_o.

The system S_o is solidary with the considered body and moves with speed v with respect to S in the direction of the versor x.

The reference body has a mass that we will call m with respect to S and m_o with respect to S_o, which is the proper system of the body as said.

In Fig.1 the perspective is simultaneously represented, from the fourth dimension, of both points of view.

For system S, the mass m associated with point P is initially at O and finally arrives at P'. From the point of view of So, the initial position of the mass m_o coincides with P_o and the final position with P'.

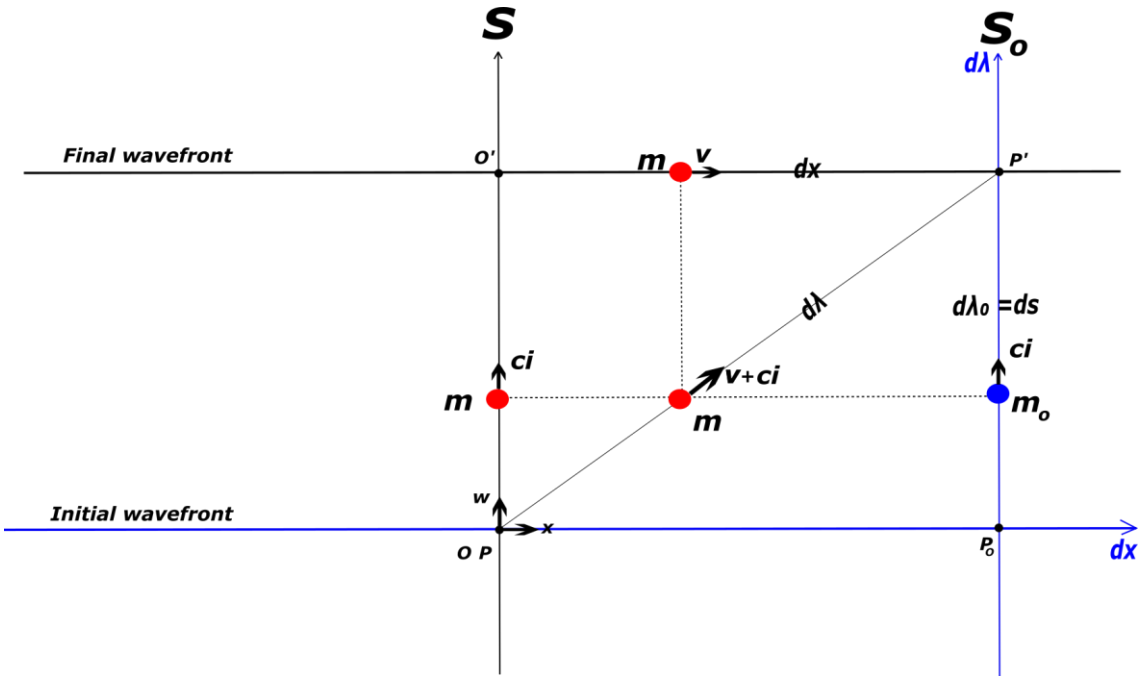

Figure 1

From the point of view of the system S, the point of mass m traveled dx according to the versor x and $d\lambda_0=ds$ according to the versor w.

The momentum according to S, which we will call **M**, is therefore:

$$M = mv\vec{x} + mci\vec{w} \qquad (1)$$

Where $i=\sqrt{-1}$ is the imaginary unit and c is the speed of light.

While the momentum according to S_o, which we will call M_o, is:

$$M_o = m_o c i \vec{w} \tag{2}$$

2. Relationship between m and m_o :

To deduce the relationship between m and m_o, we will return to the point of view raised in the previous work. In it, a geometry was introduced, which operationally responds to the physical evolution of a three-dimensional wave front, whose radius grows at the speed of light in a four-dimensional Cartesian space.

The origin of the electrostatic force between two charges was attributed to the electromagnetic attraction between them, both traveling at the speed of light in parallel trajectories in the fourth dimension.

For the corresponding calculations, remember that the value of the speed of the charges was replaced by ci, where c is the speed of light and i is the imaginary unit.
Likewise, an alternative procedure was suggested to calculate the vector product of two four-dimensional vectors, to express the deduction of said formula.

Following this line of thought, we will calculate the modul of vectors M and M_o, taking into account that we will not treat them as complex numbers but rather define their magnitudes as follows:

$$|M| = \sqrt{(mv)^2 + (mci)^2} \tag{3}$$

As long as:

$$|Mo| = \sqrt{(m_o c i)^2} \tag{4}$$

Unlike the calculation of the module of complex numbers, the imaginary unit multiply the component according to the versor w.

Therefore, if in these conditions we use the relativistic postulate that the laws of physics remain invariant in inertial reference frames, we can affirm that the module of the momentum is the same with respect to the systems S and S_o.

That is to say:

$$|M| = \sqrt{(mv)^2 + (mci)^2} = |M_o| = \sqrt{(m_o c i)^2} \tag{5}$$

$$|M|^2 = \left(\sqrt{(mv)^2 + (mci)^2} \right)^2 = |M_o|^2 = \left(\sqrt{(m_o c i)^2} \right)^2 \tag{6}$$

$$(mv)^2 + (mci)^2 = (m_o ci)^2 \tag{7}$$

$$m^2v^2 - m^2c^2 = -m_o{}^2c^2 \tag{8}$$

$$m^2\left(v^2 - c^2\right) = -m_o{}^2c^2 \tag{9}$$

$$m^2\left(c^2 - v^2\right) = m_o{}^2c^2 \tag{10}$$

$$m^2\left(1 - \frac{v^2}{c^2}\right) = m_o{}^2 \tag{11}$$

$$m = \frac{m_o}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{12}$$

Which is the well-known relationship established by Special Relativity.

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HYPERSPHERIC WAVE FRONT III
Vector Product in Hyperspace

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ABSTRACT:

In the original January 2022 essay: HYPERSPHERIC WAVEFRONT, three arguments were provided that are compatible with the reinterpretation of our three-dimensional universe, as a hyperspherical wavefront, expanding in a fourth linear dimension, at the speed of light.

In the following work of November 2022: HYPERSPHERIC WAVE FRONT II, the implications in relation to the mass of the bodies were considered.

In the present exposure: HYPERSPHERIC WAVE FRONT III, it is intended to develop a mathematically valid procedure to calculate the vector product in the fourth dimension, and thus formally justify the third argument of the original essay.

September 30, 2024

1. Nature of Hyperspace:

As in the previous works cited, we are considering a four-dimensional Cartesian hyperspace model.

The location of its points P is determined by choosing a point O, origin of the coordinate axis system Xo, Yo, Zo and Wo, which are associated respectively, to a basis of four orthonormal vectors, which we will call **io jo ko wo**.

Based on Linear Algebra, we know that the field of real numbers allows us to represent the coordinates of all sets of three linearly independent vectors belonging to ordinary three-dimensional space.

In this case, since hyperspace has four dimensions, in order to represent the coordinates of all the sets of four linearly independent vectors belonging to it, it is necessary to introduce the field of complex numbers.

In the proposed model, the coordinates of any point P are expressed with three real numbers x_o, y_o, z_o according to the **io jo ko** versors respectively, and a pure imaginary number $w_o i$ where w_o belongs to the field of real numbers and i is the imaginary unit, $i = \sqrt{-1}$.

The coordinates of the points P of hyperspace can then be expressed in the form:

Equation 1

$$P(x_o,y_o,z_o,w_o i) = x_o\vec{\mathbf{i}}_o + y_o\vec{\mathbf{j}}_o + z_o\vec{\mathbf{k}}_o + w_o i\vec{\mathbf{w}}_o$$

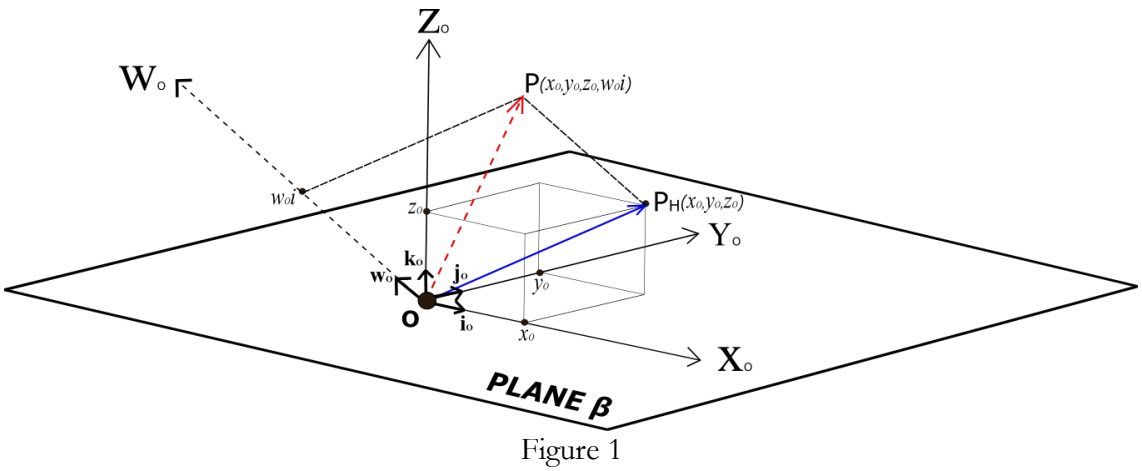

Figure 1

Figure 1 represents a point P belonging to the fourth dimension. The three axes Xo, Yo, Zo and their associated coordinates x_o, y_o, z_o, represent ordinary three-dimensional space, which we will refer to hereafter interchangeably as hyperplane, while the coordinate $w_o i$, on the axis Wo, represents the coordinate of P in the fourth spatial dimension.

We call β the plane that contains the Xo and Yo axes.

The representation in Figure 1 of the Wo axis, inclined with a dotted line, responds to the limitations of incorporating the fourth dimension into a traditional perspective, created for the third dimension. It is widely used in other works referring to the fourth dimension.

In Figure 2, point PH is the projection of point P on the hyperplane formed by the third dimension.

The three dimensions of ordinary space are "flattened" so that they can be represented by the Hyperplane in a traditional perspective.

In both figures, the vector $\overrightarrow{OP}$ is represented in red and the vector $\overrightarrow{OP_H}$ is represented in blue..

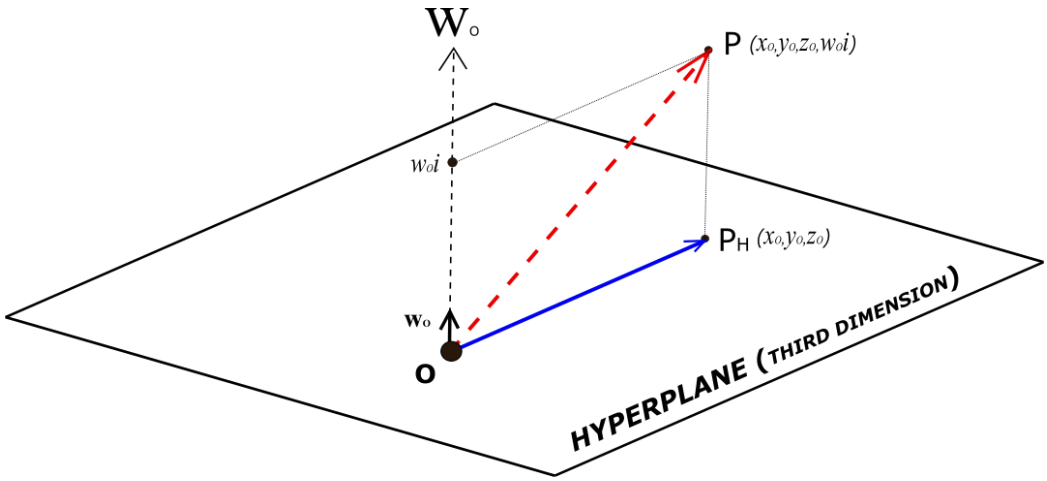

Figure 2

2. Vector product in Hyperspace:

2.1 Problem statement.

Let us now consider two vectors **u** and **v** in hyperspace, their coordinates in **io jo ko wo** are:

Equation 2

$$\mathbf{u} = u_{01}\vec{\mathbf{i}}_0 + u_{02}\vec{\mathbf{j}}_0 + u_{03}\vec{\mathbf{k}}_0 + u_{04}\vec{\mathbf{w}}_0 = (u_{01}, u_{02}, u_{03}, u_{04})$$

Equation 3

$$\mathbf{v} = v_{01}\vec{\mathbf{i}}_0 + v_{02}\vec{\mathbf{j}}_0 + v_{03}\vec{\mathbf{k}}_0 + v_{04}\vec{\mathbf{w}}_0 = (v_{01}, v_{02}, v_{03}, v_{04})$$

We remember that the coordinates u_{04} and v_{04} are pure imaginary numbers:

Equation 4

$$u_{04} = ui$$

Equation 5

$$v_{04} = vi$$

Where u and v are real numbers and:

Equation 6

$$i = \sqrt{-1}$$

The vector product is invariant with respect to changes in coordinates, so we are going to choose a coordinate system, referred to a base $\mathbf{i\ j\ k\ w}$ of origin also the point O, which simplifies our calculations, and in which the vectors $\mathbf{u}$ and $\mathbf{v}$ will be expressed:

Equation 7

$$\mathbf{u} = (u_1, u_2, u_3, u_4)$$

Equation 8

$$\mathbf{v} = (v_1, v_2, v_3, v_4)$$

In Figure 3, the vectors $\mathbf{u}$ and $\mathbf{v}$ are indicated and also the vectors $\mathbf{u_H}$ and $\mathbf{v_H}$, which are the projections of the vectors $\mathbf{u}$ and $\mathbf{v}$ in the third dimension or hyperplane.

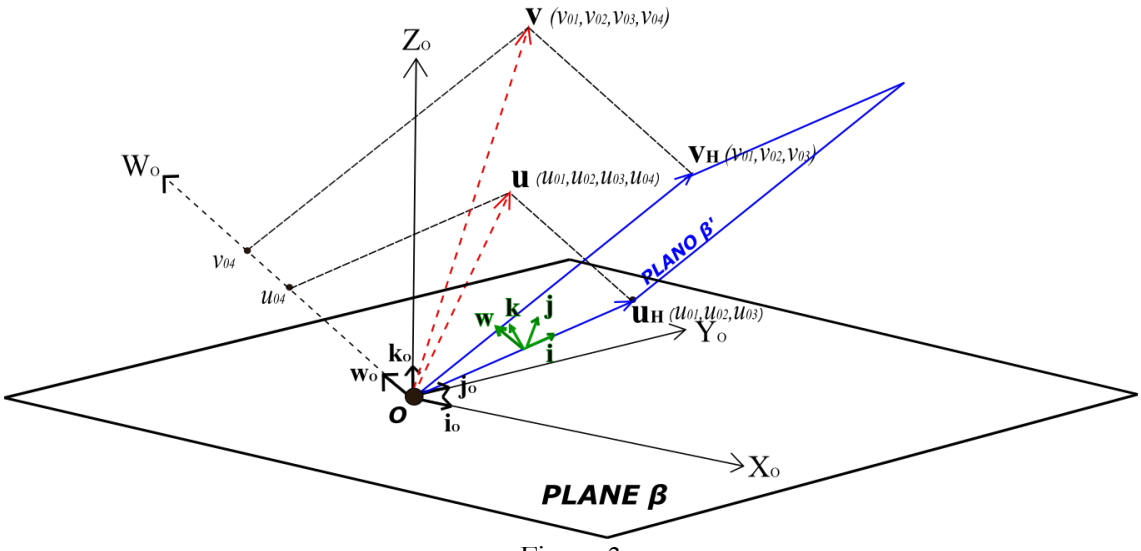

Figure 3

We will choose a transformation that leaves the $\mathbf{w}$ versor invariant, that is, $\mathbf{w} \equiv \mathbf{w_o}$, while the three axes $\mathbf{i\ j\ k}$ are chosen such that the plane defined by the vectors $\mathbf{u_H}$ and $\mathbf{v_H}$, which we will call β', coincides with the plane defined by the X and Y axes.
In addition, the vector $\mathbf{i}$ is chosen aligned with the vector $\mathbf{u_H}$, as shown in Figure 3.

As indicated, the origin of the base **i j k w** is point O, but to facilitate graphical interpretation in Figure 3 it is shown displaced from the origin.

By virtue of the proposed axis changes, we will have to:

Equation 9

$$u_1 = |\mathbf{u}_H|$$

Equation 10

$$u_2 = 0$$

Equation 11

$$u_3 = 0$$

Besides:

Equation 12

$$v_3 = 0$$

Therefore, expressing both vectors in terms of their coordinates in the basis **i j k w**, we have:

Equation 13

$$\mathbf{u} = (u_1, 0, 0, u_4)$$

Equation 14

$$\mathbf{v} = (v_1, v_2, 0, v_4)$$

In Figure 4, the vectors **u** and **v** represented in the basis **i j k w**, can be visualized.

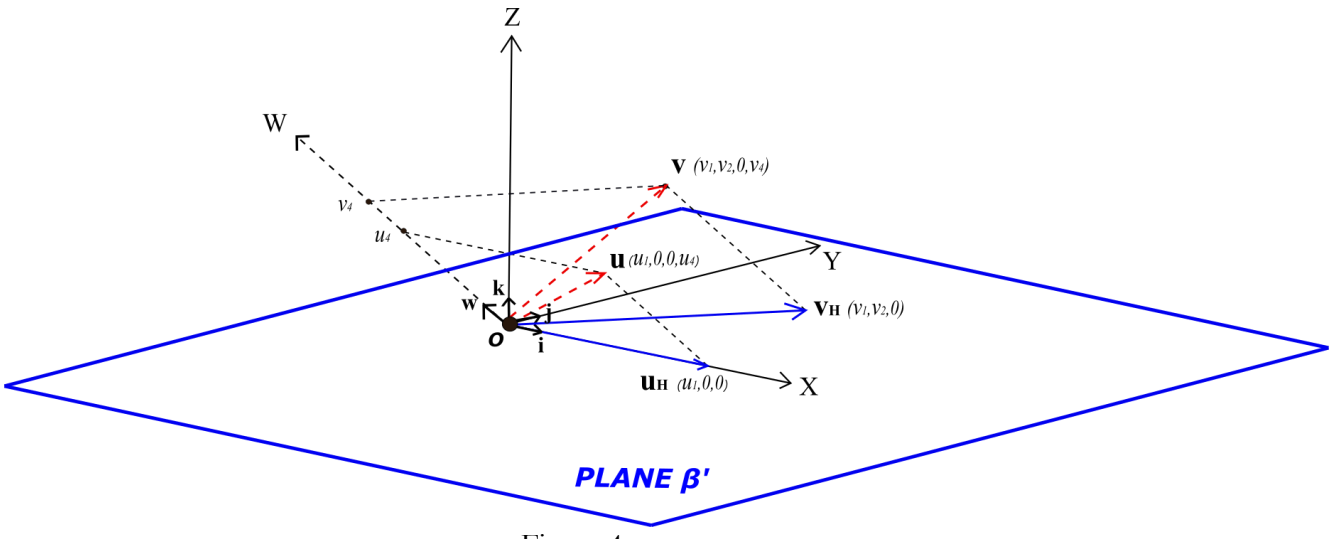

Figure 4

We must remember that our objective is to find a vector that we will call **ω**, which is the result of performing the vector product of **u** and **v** .

Equation 15

$$\boldsymbol{\omega} = \mathbf{u} \times \mathbf{v}$$

We express **ω** in terms of its coordinates:

Equation 16

$$\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3, \omega_4)$$

Vector **ω** must have as its origin the point O, and must be normal to the plane that contains the vectors **u** and **v**, which we will call ε.

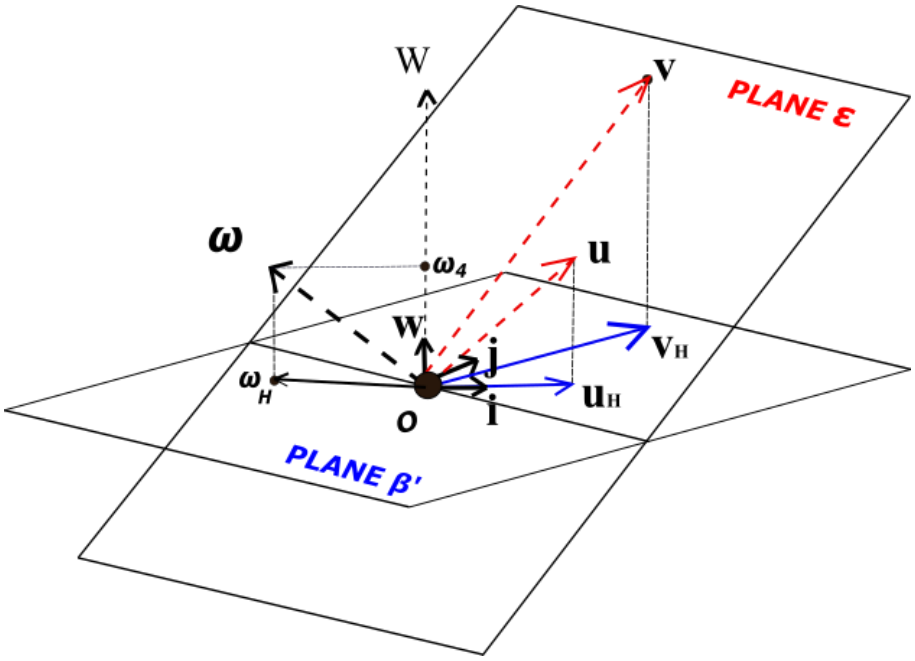

Figure 5

Let's look at Figure 5, which shows the plane defined by the vectors **uH** and **vH**, which we call β´, and the plane determined by the vectors **u** and **v**, that is ε .
The intersection of both planes passes through the origin O.

The coordinate of the vector **ω** according to **w**, that is ω4, and its projection on the plane β´, that is $\boldsymbol{\omega}_H$, are also indicated.

Equation 17

$$\boldsymbol{\omega}_H = (\omega_1, \omega_2, \omega_3)$$

But let us observe that in the chosen basis **i j k**, the coordinates according to **k** of any point or vector belonging to the plane β´, are equal to zero.

We therefore arrive at the important conclusion that $\omega_3=0$, that is the vector $\boldsymbol{\omega}_H$, is of the form:

Equation 18

$$\boldsymbol{\omega}_H = (\omega_1, \omega_2, 0)$$

Therefore, if we express the vector $\boldsymbol{\omega}$ in the system $\mathbf{i}\ \mathbf{j}\ \mathbf{k}\ \mathbf{w}$ in general form, we will have:

Equation 19

$$\boldsymbol{\omega} = (\omega_1, \omega_2, 0, \omega_4)$$

2.2 Proposed calculation procedure.

From the above, our problem is to find the vector $\boldsymbol{\omega}$ (of which we know the component $\omega_3=0$), which must be normal to the vectors $\mathbf{u}$ and $\mathbf{v}$ and also, by definition of vector product, its module must satisfy the equality $|\omega| = |u||v| sen\theta$, where θ is the angle formed by both vectors.

The proposed procedure is the same as that intuitively suggested in the original essay of this trilogy: HYPERSPHERIC WAVE FRONT, paragraph 3.

We initially consider the following equality:

Equation 20

$$\boldsymbol{\omega} = \mathbf{u} \times \mathbf{v} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{w} \\ u_1 & 0 & 0 & u_4 \\ v_1 & v_2 & 0 & v_4 \end{bmatrix}$$

The 3x3 sub-determinants that are obtained by eliminating a column and that do not involve the base vector $\mathbf{k}$ are solved, since we know that $\omega_3=0$.

Therefore, the only determinant to solve is:

Equation 21

$$\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{w} \\ u_1 & 0 & u_4 \\ v_1 & v_2 & v_4 \end{bmatrix} = -v_2 u_4 \vec{\mathbf{i}} + (-u_1 v_4 + v_1 u_4) \vec{\mathbf{j}} + u_1 v_2 \vec{\mathbf{w}}$$

Therefore the vector $\boldsymbol{\omega}$ produced by the proposed procedure has components:

Equation 22

$$\boldsymbol{\omega} = (-v_2 u_4, - u_1 v_4 + v_1 u_4, 0, u_1 v_2)$$

To facilitate the visualization of the corresponding calculations in the developed procedure, Table 1 is introduced:

Table 1

	i	**j**	**k**	**w**
u	u_1	0	0	u_4
v	v_1	v_2	0	v_4
$\boldsymbol{\omega}$	$-v_2 u_4$	$- u_1 v_4 + v_1 u_4$	0	$u_1 v_2$

<u>2.3</u> Validation of the procedure.

For the vector $\boldsymbol{\omega}$ obtained to be valid, we must check that two conditions are met, which we will call 1 and 2:

1) Normality: If $\boldsymbol{\omega}$ is normal to **u** and **v** vectors, the scalar product with each of them must be zero:

Equation 23

$$\mathbf{u} \cdot \boldsymbol{\omega} = (u_1, 0, 0, u_4) \cdot (-v_2 u_4, - u_1 v_4 + v_1 u_4, 0, u_1 v_2) = -u_1 v_2 u_4 + u_1 v_2 u_4 = 0$$

Equation 24

$$\mathbf{v} \cdot \boldsymbol{\omega} = (v_1, v_2, 0, v_4) \cdot (-v_2 u_4, - u_1 v_4 + v_1 u_4, 0, u_1 v_2) = -v_1 v_2 u_4 - u_1 v_2 v_4 + v_1 v_2 u_4 + u_1 v_2 v_4 = 0$$

What is verified.

2) Value of the module: The module of $\boldsymbol{\omega}$ must satisfy the equality that arises from the definition of vector product:

Equation 25

$$|\boldsymbol{\omega}| = |\mathbf{u}||\mathbf{v}| sen\, \theta$$

We will calculate both sides of Equation 25 independently.

By Equation 22, the module of $\boldsymbol{\omega}$ is:

Equation 26

$$|\boldsymbol{\omega}| = \sqrt{v_2^2 u_4^2 + u_1^2 v_4^2 - 2 u_1 v_4 v_1 u_4 + v_1^2 u_4^2 + u_1^2 v_2^2}$$

On the right side of Equation 25, we will calculate its three factors.

As $\mathbf{u} = (u_1,0,0,u_4)$, its module is:

Equation 27

$$|\mathbf{u}| = \sqrt{u_1^2 + u_4^2}$$

As $\mathbf{v} = (v_1,v_2,0,v_4)$, its module is:

Equation 28

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_4^2}$$

To calculate $\operatorname{sen}\theta$, we will use the definition of the scalar product of $\mathbf{u}$ and $\mathbf{v}$, which must satisfy the equality:

Equation 29

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$$

Equation 30

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$

Equation 31

$$\mathbf{u} \cdot \mathbf{v} = (u_1,0,0,u_4) \cdot (v_1,v_2,0,v_4) = u_1v_1 + u_4v_4$$

Substituting in Equation 30 what is expressed in Equations 27, 28 and 31, we have that:

Equation 32

$$\cos\theta = \frac{u_1v_1 + u_4v_4}{\sqrt{u_1^2 + u_4^2}\sqrt{v_1^2 + v_2^2 + v_4^2}}$$

Using the trigonometric expression:

Equation 33

$$sen^2\theta + \cos^2\theta = 1$$

Equation 34

$$sen\theta = \sqrt{1 - \cos^2\theta}$$

And replacing the expression of $\cos\theta$ of Equation 32 in Equation 34:

Equation 35

$$sen\theta = \sqrt{1 - \frac{(u_1 v_1 + u_4 v_4)^2}{(u_1^2 + u_4^2)(v_1^2 + v_2^2 + v_4^2)}}$$

Equation 36

$$sen\theta = \sqrt{\frac{(u_1^2 + u_4^2)(v_1^2 + v_2^2 + v_4^2) - (u_1 v_1 + u_4 v_4)^2}{(u_1^2 + u_4^2)(v_1^2 + v_2^2 + v_4^2)}}$$

Equation 37

$$sen\theta = \sqrt{\frac{u_1^2 v_2^2 + u_1^2 v_4^2 + u_4^2 v_1^2 + u_4^2 v_2^2 - 2u_1 v_1 u_4 v_4}{(u_1^2 + u_4^2)(v_1^2 + v_2^2 + v_4^2)}}$$

Equation 38

$$|\mathbf{u}||\mathbf{v}|sen\theta = \sqrt{u_1^2 + u_4^2}\sqrt{v_1^2 + v_2^2 + v_4^2}\sqrt{\frac{u_1^2 v_2^2 + u_1^2 v_4^2 + u_4^2 v_1^2 + u_4^2 v_2^2 - 2u_1 v_1 u_4 v_4}{(u_1^2 + u_4^2)(v_1^2 + v_2^2 + v_4^2)}}$$

Simplifying and rearranging terms in Equation 38 gives us:

Equation 39

$$|\mathbf{u}||\mathbf{v}|sen\theta = \sqrt{v_2^2 u_4^2 + u_1^2 v_4^2 - 2u_1 v_4 v_1 u_{4+} v_1^2 u_4^2 + u_1^2 v_2^2}$$

And comparing Equations 26 and 39, we arrive at the equality of both members of Equation 25, with which condition 2 is fulfilled.

3. Vector Product applied to the third argument of the original essay of January 2022 "Hyperspheric Wave Front":

The central hypothesis of this work, presented three arguments compatible with the reinterpretation of our three-dimensional material universe, as a hyperspherical wave front that expands in a fourth linear dimension at the speed of light.

Due to the magnitude of its radius, it can be assimilated at certain scales to a hyperplane, which by definition is also three-dimensional.

This is done in a similar way to when working in two dimensions and we consider that a spherical surface, also under appropriate conditions, can be approximated in the vicinity of a point to a plane.

It is precisely the third argument, that proposed interpreting the identity of the electrostatic force between two electrical charges at relative rest, with the magnetic force between those same charges, considering that by belonging to the hyperspherical wave front, both charges travel at the speed of light in the direction of the fourth dimension.

In Figure 6, assuming this hypothesis, we must consider that the hyperplane moves in the direction of **w** at the speed of light c, we will also assume that both charges are positive and of equal value.

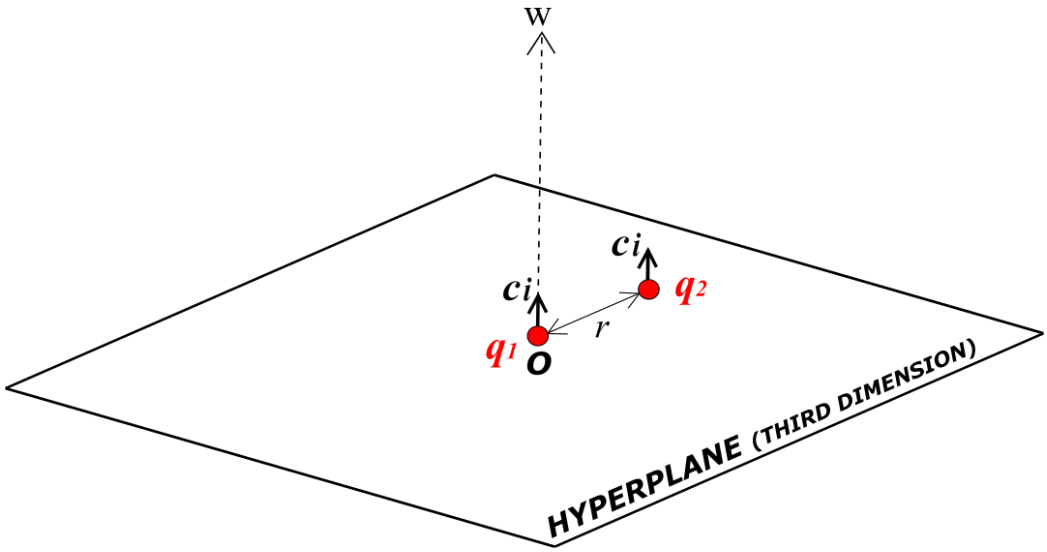

Figure 6

In Figure 7 we will consider these charges viewed from the third dimension, referenced in the **i j k w** system described in 2.1, for which we take Figure 4 as a reference.
We will locate both charges in the Plane β´(defined by the X and Y axes), with one charge at the origin O and the other on the X axis at a distance *r*.

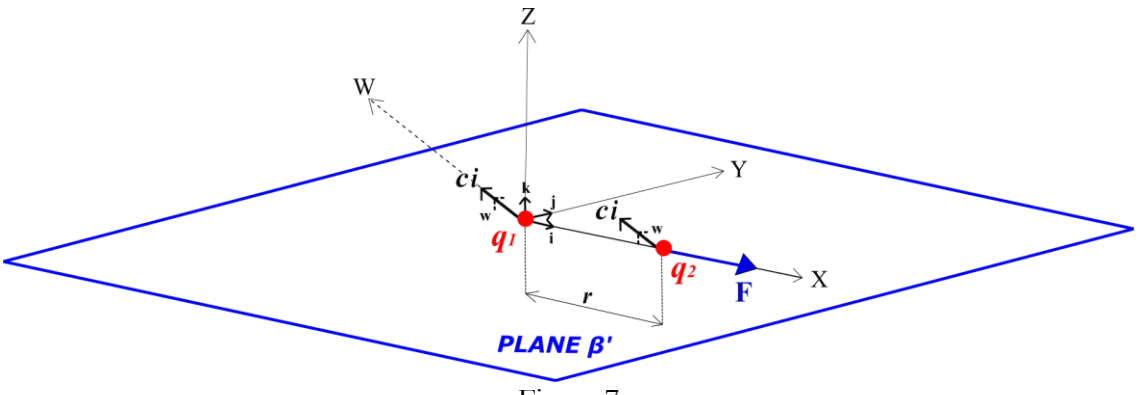

Figure 7

If we apply the Biot-Savart law to find the magnetic force **F** that the charge q_1 exerts on q_2, we have that:

Equation 40

$$\mathbf{F} = \frac{\mu_o}{4\pi} \frac{q_1 q_2 \mathbf{V}_2 \times (\mathbf{V}_1 \times (\mathbf{r}_2 - \mathbf{r}_1))}{|\mathbf{r}_2 - \mathbf{r}_1|^3}$$

The considered charges q_1 and q_2 are of equal value and sign, with q_1 the charge located at the origin, and $\mathbf{V_1}$ and $\mathbf{V_2}$ their parallel velocities and of equal magnitude, with $\mathbf{r_2}$ and $\mathbf{r_1}$ being their position vectors.

Therefore:

Equation 41

$$q_2 = q_1 = q$$

Equation 42

$$\mathbf{V}_2 = \mathbf{V}_1 = ci\vec{\mathbf{w}}$$

Equation 43

$$(\mathbf{r}_2 - \mathbf{r}_1) = r\vec{\mathbf{i}}$$

Equation 44

$$|\mathbf{r}_2 - \mathbf{r}_1|^3 = r^3$$

Rearranging and substituting the above in Equation 40, gives us:

Equation 45

$$\vec{\mathbf{F}} = \frac{\mu_o q^2}{4\pi r^3} \mathbf{V}_2 \times (\mathbf{V}_1 \times (\mathbf{r}_2 - \mathbf{r}_1))$$

To calculate the two vector products, we will apply the previously developed procedure.

We first calculate $\mathbf{V}_1 \times (\mathbf{r}_2 - \mathbf{r}_1)$, by looking at Table 1 and replacing the coordinates of both axes in it, we obtain Table 2:

Equation 46

$$\mathbf{V}_1 = (0,0,0,ci)$$

Equation 47

$$(\mathbf{r}_2 - \mathbf{r}_1) = (r,0,0,0)$$

We calculate the product:

Equation 48

$$\mathbf{V}_1 \times (\mathbf{r}_2 - \mathbf{r}_1) = (0,0,0,ci) \times (r,0,0,0)$$

Table 2

	i	**j**	**k**	**w**
$\mathbf{V}_1$	0	0	0	ci
$(\mathbf{r}_2 - \mathbf{r}_1)$	r	0	0	0
$\mathbf{V}_1 \times (\mathbf{r}_2 - \mathbf{r}_1)$	0	rci	0	0

We thus obtain the first vector product :

Equation 49

$$\mathbf{V}_1 \times (\mathbf{r}_2 - \mathbf{r}_1) = (0,rci,0,0)$$

On the other hand:

Equation 50

$$\mathbf{V}_2 = (0,0,0,ci)$$

We calculate the other vector product, using Table 3:

Equation 51

$$\mathbf{V}_2 \times \mathbf{V}_2 \times (\mathbf{r}_2 - \mathbf{r}_1) = (0,0,0,ci) \times (0,rci,0,0)$$

Table 3

	i	**j**	**k**	**w**
$\mathbf{V}_2$	0	0	0	ci
$\mathbf{V}_1 \times (\mathbf{r}_2 - \mathbf{r}_1)$	0	rci	0	0
$\mathbf{V}_2 \times \mathbf{V}_1 \times (\mathbf{r}_2 - \mathbf{r}_1)$	$-rc^2i^2$	0	0	0

The result is then:

Equation 52

$$\mathbf{V}_2 \times \mathbf{V}_2 \times (\mathbf{r}_2 - \mathbf{r}_1) = (0,0,0ci) \times (0,rci,0,0) = (-rc^2i^2,0,0,0) = rc^2\vec{\mathbf{i}}$$

Substituting in Equation 45 we have that:

Equation 53

$$\mathbf{F} = \frac{\mu_o q^2}{4\pi r^3} \mathbf{V}_2 \times (\mathbf{V}_1 \times (\mathbf{r}_2 - \mathbf{r}_1)) = \frac{\mu_o q^2}{4\pi r^3} rc^2\vec{\mathbf{i}}$$

Finally simplifying we obtain Equation 54, which is the well-known expression of the electrostatic repulsion between equal charges of the same sign.

Note that in this case the force **F** is repulsive, contrary to the case of the magnetic force between the same charges, if they were moving with equal speeds in the third dimension.

Equation 54

$$\mathbf{F} = \frac{\mu_o q^2 c^2}{4\pi r^2} \vec{\mathbf{i}}$$

—°—

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HYPERSPHERIC WAVE FRONT IV
Complex Hyperspace $\mathbb{C}^7$

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Abstract

The *Hyperspheric Wave Front* theory was presented in three previous works published on ResearchGate:

-January 2022: *Hyperspheric Wave Front*.

-November 2022: *Hyperspheric Wave Front II: Additional considerations related to mass*.

-September 2024: *Hyperspheric Wave Front III: Vector Product in Hyperspace*.

These works developed the idea of reinterpreting our three dimensional universe as a hyperspherical wavefront expanding at the speed of light along a fourth linear dimension.
In those studies, a four dimensional space, three real dimensions and one imaginary, was considered as the contingent framework of our physical reality, according to this theory.

This representation presents mathematical limitations regarding the calculation of vector products and other aspects, such as computing the modulus of vectors with complex coordinates under certain conditions. These issues were addressed in those works from a conceptual and heuristic standpoint.

In the present essay, we propose reinserting the proposed four dimensional model, three real axes and one imaginary axis, with the same properties and characteristics previously described, into a seven dimensional contingent complex vector space: the Complex Hyperspace $\mathbb{C}^7$.

The properties of such a space have been widely studied and described within the field of Linear Algebra, particularly in the study of *n*-dimensional Complex Vector Spaces.

We argue that the present work reinforces the conceptual validity presented in the first three studies, while framing them within a formal mathematical reality.

August 24, 2025

1. Introduction:

In the work published in September 2024, *Hyperspheric Wave Front III*, a mathematically operational, though not formally rigorous, procedure was developed, based on the approach proposed in the original essay, to compute the vector products outlined in the third argument of the first essay.

This procedure was necessary to carry out the calculations of the proposed hypothesis, which identifies the electrostatic forces between two charges at relative rest in the third dimension with the magnetic origin forces generated by their displacement at the speed of light in the fourth dimension.

Furthermore, according to the Theory of Normed Algebras and alternative systems such as the Octonions, the bilinear and antisymmetric vector product, with properties similar to that of $\mathbb{R}^3$, namely producing a vector orthogonal to both input vectors, exists with non-trivial results only in real dimensions three and seven, that is, $\mathbb{R}^3$ and $\mathbb{R}^7$.

The works published by John C. Baez [21] [22], Conway and Smith [23], Tevian Dray and Corinne A. Manogue [24], among others, show that it is possible to extend the vector product of two vectors to a seven dimensional complex vector space, denoted $\mathbb{C}^7$, where the scalars defining the vector coordinates are complex numbers.

Since our goal is to work within a vector space capable of representing three real axes and one imaginary axis, and we require the formal definition of the vector product for two vectors belonging to such a space, it follows that the only viable option capable of satisfying these requirements is the **Complex Hyperspace** $\mathbb{C}^7$.

2. Complex Hyperspace $\mathbb{C}^7$:

Since $\mathbb{C}^7$ is a vector space over the complex numbers, it has dimension 7, and is closed under vector addition and scalar multiplication by elements of $\mathbb{C}$.

As a vector space over the real numbers, each complex component is equivalent to two real components; therefore, $\mathbb{C}^7$ is isomorphic to $\mathbb{R}^{14}$.

A canonical basis is given by:

Equation 1

$$\mathbf{e}_1 = (1,0,0,0,0,0,0), \mathbf{e}_2 = (0,1,0,0,0,0,0), \ldots . \mathbf{e}_7 = (0,0,0,0,0,0,1)$$

Any vector can be expressed as a linear combination of this basis with complex coefficients.

Thus, a vector $\mathbf{u}$ in $\mathbb{C}^7$ can be written in general form as:

Equation 2

$$\mathbf{u} = (u_1, u_2, u_3, u_4, u_5, u_6, u_7) = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3 + u_4\mathbf{e}_4 + u_5\mathbf{e}_5 + u_6\mathbf{e}_6 + u_7\mathbf{e}_7$$

where the coordinates are complex numbers.

The inner product (Hermitian inner product, "scalar product") of two vectors $\mathbf{u}$ and $\mathbf{v}$ belonging to $\mathbb{C}^7$, denoted as $\mathbf{u} \bullet \mathbf{v}$, is defined by:

Equation 3

$$\mathbf{u} = (u_1, u_2, u_3, u_4, u_5, u_6, u_7) = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3 + u_4\mathbf{e}_4 + u_5\mathbf{e}_5 + u_6\mathbf{e}_6 + u_7\mathbf{e}_7$$

Equation 4

$$\mathbf{v} = (v_1, v_2, v_3, v_4, v_5, v_6, v_7) = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3 + v_4\mathbf{e}_4 + v_5\mathbf{e}_5 + v_6\mathbf{e}_6 + v_7\mathbf{e}_7$$

Equation 5

$$\mathbf{u} \bullet \mathbf{v} = \sum_{k=1}^{7} \overline{u}_k v_k,$$

where $\overline{u}_k$ is the complex conjugate of u_k.

Therefore, the modulus of a vector $\mathbf{u}$ is:

Equation 6

$$\left\|\mathbf{u}\right\| = \sqrt{\overline{\mathbf{u} \bullet \mathbf{u}}}$$

The role played in the previous works by the basis $\mathbf{i}\ \mathbf{j}\ \mathbf{k}\ \mathbf{w}$, where $\mathbf{i}\ \mathbf{j}\ \mathbf{k}$ represents the triple in $\mathbb{R}^3$ and the unit vector $\mathbf{w}$ corresponds to the imaginary axis perpendicular to it, and consequently belonging to the fourth dimension, with respect to which the third dimension expands at the speed of light, is now expressed as:

Equation 7

$$\mathbf{e}_1 = \mathbf{i}$$

Equation 8

$$\mathbf{e}_2 = \mathbf{j}$$

Equation 9

$$\mathbf{e}_3 = \mathbf{k}$$

Equation 10

$$\mathbf{e}_4 = \mathbf{w}$$

3. Vector Product in $\mathbb{C}^7$:

The multiplication table presented by the previously cited authors, derived from the octonion multiplication tables, is as follows:

×	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_5$	$\mathbf{e}_6$	$\mathbf{e}_7$
$\mathbf{e}_1$	0	e_3	$-e_2$	e_5	$-e_4$	e_7	$-e_6$
$\mathbf{e}_2$	$-e_3$	0	e_1	e_6	e_7	$-e_4$	$-e_5$
$\mathbf{e}_3$	e_2	$-e_1$	0	e_7	$-e_6$	e_5	$-e_4$
$\mathbf{e}_4$	$-e_5$	$-e_6$	$-e_7$	0	e_1	e_2	e_3
$\mathbf{e}_5$	e_4	$-e_7$	e_6	$-e_1$	0	$-e_3$	e_2
$\mathbf{e}_6$	$-e_7$	e_4	$-e_5$	$-e_2$	e_3	0	e_1
$\mathbf{e}_7$	e_6	e_5	e_4	$-e_3$	$-e_2$	$-e_1$	0

Table 1

4. Calculation of the Vector Product of the Third Argument in $\mathbb{C}^7$:

As stated in the Introduction, we aim to calculate the following expression:

Equation 11

$$\mathbf{V}_2 \times (\mathbf{V}_1 \times (\mathbf{r}_2 - \mathbf{r}_1))$$

This expression contains two vector products, which we calculate within the framework of Complex Hyperspace $\mathbb{C}^7$, where the vectors involved are expressed as:

Equation 12

$$\mathbf{V}_1 = (0,0,0,ci,0,0,0) = ci\mathbf{e}_4$$

Equation 13

$$(\mathbf{r}_2 - \mathbf{r}_1) = (r,0,0,0,0,0,0) = r\mathbf{e}_1$$

Applying Table 1, we obtain:

Equation 14

$$\mathbf{V}_1 \times (\mathbf{r}_2 - \mathbf{r}_1) = (0,0,0,ci,0,0,0) \times (r,0,0,0,0,0,0) = ci\mathbf{e}_4 \times r\mathbf{e}_1 = -rci\mathbf{e}_5$$

Thus, the first desired vector product is:

Equation 15

$$\mathbf{V}_1 \times (\mathbf{r}_2 - \mathbf{r}_1) = -rci\mathbf{e}_5$$

On the other hand:

Equation 16

$$\mathbf{V}_2 = (0,0,0,ci,0,0,0) = ci\mathbf{e}_4$$

We then calculate the second vector product, again using Table 1:

Equation 17

$$\mathbf{V}_2 \times (\mathbf{V}_1 \times (\mathbf{r}_2 - \mathbf{r}_1)) = (0,0,0,ci,0,0,0) \times (0,0,0,0,-rci,0,0) = ci\mathbf{e}_4 \times -rci\mathbf{e}_5 = -rc^2 i^2 \mathbf{e}_1$$

Recalling that $i = \sqrt{-1}$, the final result coincides with the expression presented in *Hyperspheric Wave Front III:*

Equation 18

$$\mathbf{V}_2 \times (\mathbf{V}_1 \times (\mathbf{r}_2 - \mathbf{r}_1)) = rc^2 \mathbf{e}_1$$

5. Relativistic Mass in $\mathbb{C}^7$

5.1 Introduction

As previously indicated in the essay *"Hyperspheric Wave Front II: Additional considerations related to Mass"*, expressions were calculated for the momentum vectors of a particle P of mass ***m*** with respect to two coordinate systems, S and S_o, the latter being the particle's rest frame, as shown in Figure 1.

The momentum according to S is denoted **M**, whereas the momentum according to S_o is denoted $\mathbf{M}_o$.

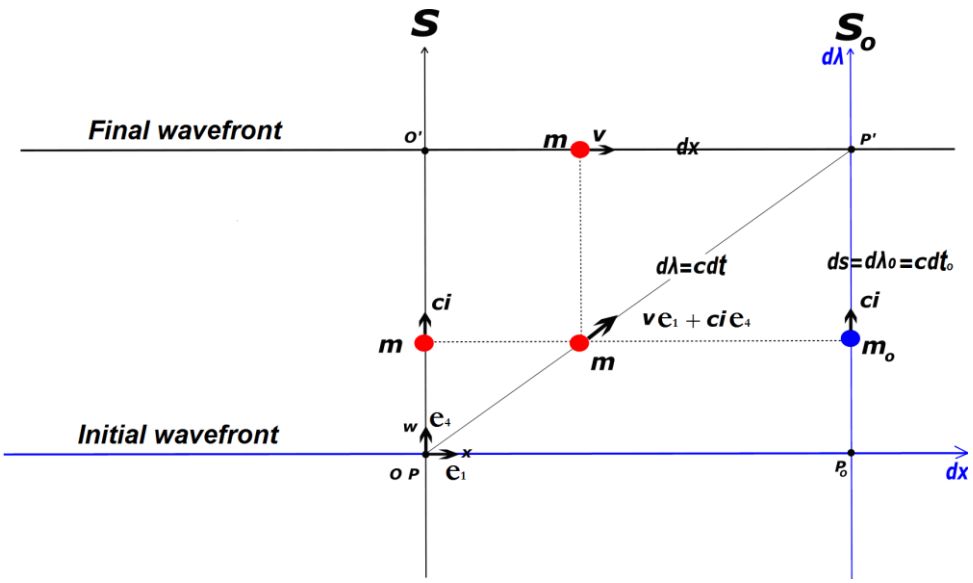

Figure 1

Expressing these momentum in $\mathbb{C}^7$, we have:

Equation 19

$$M = (mv,0,0,mci,0,0,0) = mv\mathbf{e}_1 + mci\mathbf{e}_4$$

Equation 20

$$M_o = (0,0,0,m_oci,0,0,0) = m_oci\mathbf{e}_4$$

Where $i = \sqrt{-1}$ is the imaginary unit and c is the speed of light.

In that work, the equality of the moduli of **M** and $\mathbf{M}_o$ was proposed, suggesting a particular way to calculate them, which we now demonstrate coincides with the method presented below.

5.2 Definition of the Pseudonorm of a Vector in $\mathbb{C}^7$

As seen in Equation 6, the modulus of a vector in $\mathbb{C}^7$ is defined as $\|\mathbf{u}\| = \sqrt{\bar{\mathbf{u}} \bullet \mathbf{u}}$; however, for the present purpose, we will not use it to calculate the moduli of **M** and $\mathbf{M}_o$.

Instead, we employ the **signed pseudonorm** commonly used in relativistic physics, which is an indefinite Hermitian quadratic form used in Minkowski metrics with signature (+,+,+,−), and here extended by analogy to $\mathbb{C}^7$.

Given a vector $\mathbf{u} \in \mathbb{C}^7$:

Equation 21

$$\mathbf{u} = (u_1, u_2, u_3, u_4, u_5, u_6, u_7) = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3 + u_4\mathbf{e}_4 + u_5\mathbf{e}_5 + u_6\mathbf{e}_6 + u_7\mathbf{e}_7$$

The pseudonorm thus defined is:

Equation 22

$$\|\mathbf{u}\|^2_{pseudonorm} = |u_1|^2 + |u_2|^2 + |u_3|^2 - |u_4|^2 - |u_5|^2 - |u_6|^2 - |u_7|^2$$

where, by the definition of the modulus of a complex number:

Equation 23

$$|u_k| = \sqrt{\bar{u}_k u_k}$$

And the vectors:

Equation 24

$$M = (mv, 0, 0, mci, 0, 0, 0) = mv\mathbf{e}_1 + mci\mathbf{e}_4$$

Equation 25

$$M_o = (0, 0, 0, m_o ci, 0, 0, 0) = m_o ci\mathbf{e}_4$$

The respective pseudonorms are:

Equation 26

$$\|M\|^2_{pseudonorm} = (mv)^2 - (mc)^2$$

Equation 27

$$\|M_o\|^2_{pseudonorm} = -(m_o c)^2$$

5.3 Final Expression of Mass

By the hypothesis of equality of the two moduli:

Equation 28

$$\|M\|^2_{pseudonorm} = \|M_o\|^2_{pseudonorm}$$

Equating both expressions and expanding, we obtain:

Equation 29

$$m^2v^2 - m^2c^2 = -m_o{}^2c^2$$

Equation 30

$$m^2(v^2 - c^2) = -m_o{}^2c^2$$

Equation 31

$$m^2(c^2 - v^2) = m_o{}^2c^2$$

Equation 32

$$m^2\left(1 - \frac{v^2}{c^2}\right) = m_o{}^2$$

Equation 33

$$m = \frac{m_o}{\sqrt{1 - \frac{v^2}{c^2}}}$$

This corresponds to the dependence of mass on velocity established by Special Relativity, derived here from an alternative perspective.

—°—

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